Analytic Combinatorics Exercise Sheet 5

Exercises for the session on 29/5/2017

Problem 5.1

A generating function of the form $F(z) = \Phi(z, F(z))$ is said to have characteristic function $\Phi(z, w)$, and a general formula for such a function is given by

$$[z^n]F(z) \sim \frac{\alpha}{2\sqrt{\pi}} \left(\frac{1}{r}\right)^n n^{-\frac{3}{2}}, \tag{1}$$

where r is the smallest positive real solution to the simultaneous equations

$$\Phi(r, s) = s \text{ and } \frac{\partial \Phi}{\partial w}(r, s) = 1,$$

and where

$$\alpha = \sqrt{\frac{2r\frac{\partial \Phi}{\partial z}(r, s)}{\frac{\partial^2 \Phi}{\partial w^2}(r, s)}}.$$

Let $S(z)$ denote the generating function for the class of bracketings, as defined in Problem 1.2. Find r and evaluate the right-hand-side of (1) to obtain

$$[z^n]S(z) \sim \sqrt{\frac{r}{\pi 8\sqrt{2}}} \left(\frac{1}{r}\right)^n n^{-\frac{3}{2}}.$$

Problem 5.2

Show that the centre of any tree contains at most two vertices. Show also that if the centre of a tree contains two vertices, then those vertices are neighbours.

Problem 5.3

Show that the distribution of the number of cycles of length $l \in \mathbb{N}$ in a random permutation of size n converges to a Poisson distribution of rate $1/l$.

Problem 5.4

The saddle-point bound is given by

$$[z^n]G(z) \le \frac{G(\zeta)}{\zeta^n}, \tag{2}$$

where ζ is the unique real root to $\frac{\zeta G'(\zeta)}{G(\zeta)} = n+1$.

Evaluate the right-hand-side of (2) for $G(z) = e^z$, and show that the bound obtained is higher than the true value by a factor of $\sqrt{2\pi n}$.

The saddle-point approximation formula is given by

$$[z^n]G(z) \sim \frac{G(\zeta)}{\zeta^{n+1}\sqrt{2\pi g''(\zeta)}}, \tag{3}$$

where $g(z) = \ln G(z) - (n+1)\ln z$.

Evaluate the right-hand-side of (3) for $G(z) = e^z$, and show that the expression obtained is correct.

Problem 5.5

Evaluate the right-hand-side of (2) for $G(z) = (1+z)^{2n}$, and show that the bound obtained is higher than the true value by a factor of $\sqrt{\pi n}$. Then evaluate the right-hand-side of (3), and show that the expression obtained is correct.

Problem 5.6

Let S_n denote the number of ways to partition a labelled set of size n (so, for example, $S_3 = 5$ by considering the partitions $\{1\}\{2\}\{3\}$, $\{1\}\{2,3\}$, $\{2\}\{1,3\}$, $\{3\}\{1,2\}$, and $\{1,2,3\}$), and let $S(z) = \sum_{n\ge 0} \frac{S_n}{n!} z^n$ denote the corresponding exponential generating function.

Evaluate the right-hand-side of (2) for $G(z) = S(z)$ to show

$$[z^n]S(z) \le \frac{e^{n-1}}{(\ln n)^n}$$

(you may use the fact that the solution to the equation $\zeta e^{\zeta} = n+1$ is given by $\zeta \sim \ln n - \ln\ln n$).